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Extended-order algebras

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Abstract

Extended-order algebras are defined, whose operation extends the order relation of a poset with a greatest element. Most implicative algebras, including Hilbert algebras and BCK algebras fall within this context. Several classes of extended-order algebras are considered that lead to most well known multiplicative ordered structures by means of adjunction, once the completion process due to MacNeille is applied. In particular, complete distributive extended-order algebras are considered as a generalization of complete residuated lattices, to provide a structure that suits quite well for many-valued mathematics.

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1. Introduction

First we explain motivation and scope of this paper. To this extent we establish some notation concerning mathematical concepts that we need here and shall use later on, with possible variation of some letters and symbols. A binary relation from a set X to a set Y is a subset $r \subseteq X \times Y$. A binary L-relation from X to Y is a map $\mathcal{R} : X \times Y \to L$, where L is usually assumed to be some kind of algebra, $|L| \geqslant 2$, with a distinguished element $\top \in L$. We shall always consider binary (L-)relations, so we shall sometimes omit the attribute binary. Binary L-relations have been frequently considered with L a complete residuated lattice (see [2,3,8,16,17]) and in particular L-order, L-similarity and L-equality have been studied in connection with lattice valued sets (see [2–5,8,16,23]).

The distinguished element $\top \in L$ allows to consider a surjective correspondence mapping an L-relation $\mathcal{R} : X \times Y \to L$ to the relation $r \subseteq X \times Y$ defined by the equivalence $x\ r\ y \Leftrightarrow \mathcal{R}(x, y) = \top$ and called the relation naturally associated to $\mathcal{R}$. Conversely, every L-relation which the relation r is associated to will be called an L-extension of r.

Of course a binary L-relation on L can be viewed as a binary operation in L and conversely; so any relation $r \subseteq L \times L$ in $(L, \top)$ can be extended (not univocally, unless either $|L| = 2$ or $R = L \times L$) to an operation, say $\mathcal{R} : L \times L \to L$, whose naturally associated relation, in the above sense, is r. If r is reflexive ($x\ r\ x$) and antisymmetrical

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($x\ r\ y, y\ r\ x \Rightarrow x = y$) and $\top\ r\ x$, $\forall x \in L$, then any extension $\mathcal{R}$ of r gives a kind of structure $(L, \mathcal{R}, \top)$ considered in [30,31] and called d-algebra; conversely, any d-algebra $(L, *, 0)$ determines (univocally) in the above sense a reflexive, antisymmetrical relation r in L such that $0\ r\ x$, $\forall x \in L$.

In this paper we shall consider algebras of type $(2, 0)$ whose binary operation is any extension of the order relation of a poset with a greatest element $(L \leqslant, \top)$ that we shall call weak extended-order algebras (shortly w-eo algebras): we could say that w-eo algebras are d-transitive d-algebras (those whose associated relation r is transitive) in their right form. Also w-eo algebras can be considered as quite general implication algebras: most of these in fact, including Hilbert algebras (see [9,13]) and BCK algebras (in their right form, according to notation and terminology of [26]), are special kinds of w-eo algebras. Also, residuated lattices are, with respect to their residuum and the unit of multiplication, special w-eo algebras.

Our main purpose is to provide a quite general structure, rich enough as to get a range for the so-called "many valued" or "fuzzy" mathematics: it is well known that people working in this area assume to have a complete lattice (quite frequently the unit interval) with additional properties that could be the infinite or even complete distributivity conditions, so giving at least a complete Heyting algebra, or an adjoint pair of binary operations, including an associative commutative product with unit, so giving a complete residuated lattice. Sometimes only an additional compatible multiplicative structure is considered, so giving an order-complete po-groupoid that, with more restrictive conditions such as to be a strictly two sided commutative quantale or a semi-continuous t-norm, leads to a complete residuated lattice (see the discussion at the end of Section 5).

One could say that a complete residuated lattice is "the most general rich enough" structure being used for many valued mathematics, according to the recent trend in that area (see [2–5,8,14,16–18,23,24]). However, what is beyond the completeness and the adjoint conditions in a complete residuated lattice, i.e. the additional properties of the product, tends to be sometimes removed or weakened to some extent, until they are needed. The way we propose that leads to these kind of weaker structures that suit as a range for many-valued mathematics consists in the following steps:

(1) we start with a $(2, 0)$-type algebra $(L, \rightarrow, \top)$, with operations resembling an implication and a truth value, that determine a natural ordering $\leqslant$ in L (w-eo algebra); all the conditions needed in the subsequent steps are expressed in terms of $\rightarrow$, $\top$ and $\leqslant$;
(2) then we restrict to extended-order algebras (eo algebras) that can be naturally embedded in their MacNeille completion;
(3) a simple distributivity condition on the eo algebra (giving a deo algebra) allows to get in its completion a product $\otimes$ that is left adjoint to the implication $\rightarrow$;
(4) suitable conditions, once more in terms of $\rightarrow$, $\top$ and $\leqslant$, characterize deo algebras whose completion have products with either of the usually required additional properties such as, for instance, associativity and commutativity.

Among the main features of the eo algebras, with possible further conditions, we remark the following:

- the main operations $\rightarrow$ and $\top$ determine naturally and univocally a partial order and a product in the underlying set;
- the classical Dedekind–MacNeille completion process preserves all the properties we deal with;
- being "modeled" on a sort of implication, the structure is not commutative and, even more, it is unsymmetrical, since the operation $\rightarrow$ requires distributivity conditions which are of a different kind on the left and on the right side. Instead, while dealing with a possibly non-commutative product, resembling conjunction, one would wonder in which argument he would expect or require a distributivity condition, since it could occur in the same way in both variables.

As an information we note that on one hand structures with a similar name as the ones we consider have been introduced in [15] (see also [6]): order algebras considered in [15] are univocally determined by posets and those are not w-eo algebras; in [6] instead a subclass of w-eo algebras is considered, whose elements are Hilbert algebras too. On the other hand, the weakest structures we consider (w-eo algebras) as a starting point of our development are also considered in [32], with the different name of implicative algebras, as a starting point of a path going through the main implication algebras considered in non-classical logic; the path we go along in this paper is complementary, in a sense we shall explain in the concluding section. The end points of these two paths both coincide with boolean algebras.

2. Preliminaries

We recall some well known notions on order, lattices and related topics, that will be useful all along the paper, and fix explicitly the notations we shall adopt, most of which are quite standard in recent literature.

In a pre-ordered set $(L, \leqslant)$, i.e. a set with a reflexive and transitive relation, $x \in L$ is an upper (lower, respectively) bound of a subset $S \subseteq L$ if $a \in S$ implies $a \leqslant x$ ($x \leqslant a$, respectively). We denote by $Ub_L(S)$ ($Lb_L(S)$, respectively), or by $Ub_{\leqslant}(S)$ ($Lb_{\leqslant}(S)$, respectively), or more simply by $Ub(S)$ ($Lb(S)$, respectively), the set of all upper (lower, respectively) bounds of S in the pre-ordered set $(L, \leqslant)$. Similarly, we denote by $\bigvee_L S$ and $\bigwedge_L S$, or by $\bigvee_{\leqslant} S$ and $\bigwedge_{\leqslant} S$, or more simply by $\bigvee S$ and $\bigwedge S$, the set of the least upper bounds and the set of the greatest lower bounds of S in L: of course these may be empty and in case when they are singletons $\bigvee S$ and $\bigwedge S$ denote their element, as well. $Ub(a)$, $Lb(a)$ stand for $Ub(\{a\})$, $Lb(\{a\})$ and they are also denoted by $\uparrow a$, $\downarrow a$ respectively.

We assume that a complete lattice (usually denoted by $(L, \vee, \wedge, \bot, \top)$, $(L, \leqslant)$ or L) is an ordered set, with at least two elements, containing sups and infs of arbitrary subsets $S \subseteq L$, denoted by $\bigvee S$ and $\bigwedge S$ or similarly, according to the above notation. In case of a finite subset $F \subseteq L$, small symbols can be used as follows: $\vee F$ and $\wedge F$.

An *adjunction* between ordered sets L and M, denoted by $f \dashv g$, is a pair of maps $f : L \to M$ (left adjoint to g) and $g : M \to L$ (right adjoint to f) satisfying the condition $x \leqslant g(y) \Leftrightarrow f(x) \leqslant y$, for all $x \in L$ and $y \in M$; equivalently, $f \dashv g$ if f and g are isotonic and satisfy the adjoint inequalities:

$$\text{(AD1)} \quad x \leqslant g\big(f(x)\big), \quad \forall x \in L,$$

$$\text{(AD2)} \quad f\big(g(y)\big) \leqslant y, \quad \forall y \in M.$$

It will be useful to refer to the so-called adjoint functor theorem, for ordered categories.

Theorem 1. *Let $(L, \leqslant)$ and $(M, \leqslant)$ be ordered sets. Then the following hold.*

(1) *If $f : L \to M$ and $g : M \to L$ are isotonic maps and $f \dashv g$, then, for all $A \subseteq L$ and $B \subseteq M$:*
   (a) $\bigvee A \neq \emptyset \Rightarrow f(\bigvee A) = \bigvee f(A)$,
   (b) $\bigwedge B \neq \emptyset \Rightarrow g(\bigwedge B) = \bigwedge g(B)$.
(2) *If L is a complete lattice and $f : L \to M$ preserves $\bigvee$, then the function*

$$g : M \to L, \; y \mapsto g(y) = \bigvee\big\{x \in L \mid f(x) \leqslant y\big\}$$

preserves order and it is the unique right adjoint of f.
(3) *If M is a complete lattice and $g : M \to L$ preserves $\bigwedge$, then the function*

$$f : L \to M, \; x \mapsto f(x) = \bigwedge\big\{y \in M \mid x \leqslant g(y)\big\}$$

preserves order and it is the unique left adjoint of g.

A *Galois connection* between ordered sets L, M, denoted by $[f, g]$, is a pair of maps $f : L \to M$ and $g : M \to L$ such that for any $x \in L$ and $y \in M$ the condition: $x \leqslant g(y) \Leftrightarrow y \leqslant f(x)$ is satisfied. This condition is equivalent to the assumption that f and g are antitonic and satisfy the inequalities:

$$\text{(GC1)} \quad x \leqslant g\big(f(x)\big), \quad \forall x \in L,$$

$$\text{(GC2)} \quad y \leqslant f\big(g(y)\big), \quad \forall y \in M.$$

Proposition 2. *Let $[f, g]$ be a Galois connection between posets L and M. Then, for all $A \subseteq L$:*

(1) $\bigvee A \neq \emptyset \Rightarrow f(\bigvee A) = \bigwedge f(A)$,
(2) $\bigwedge A \neq \emptyset \Rightarrow f(\bigwedge A) \in Ub(f(A))$,
(3) *for each $y \in M$, $g(y) = \bigvee\{x \in L \mid y \leqslant f(x)\}$.*

Proposition 3. *Let L be a complete lattice, M a poset and let $f: L \to M$ satisfy the condition, for every $A \subseteq L$, $f(\bigvee A) = \bigwedge f(A)$.*

If $g: M \to L$ is defined by

$$g(y) = \bigvee \{x \in L \mid y \leqslant f(x)\}, \quad \text{for all } y \in M$$

then $[f, g]$ is a Galois connection.

Example 4. If $(L, \leqslant)$ is an ordered set and $Lb, Ub: \mathcal{P}(L) \to \mathcal{P}(L)$ denote the obviously defined maps bringing all $S \subseteq L$ to $Lb(S)$ and $Ub(S)$, respectively, then $[Lb, Ub]$ is a Galois connection, also called polarity of the ordered set L.

It is clear that $[f, g]$ is a Galois connection if and only if $[g, f]$ is a Galois connection. As a consequence, the statements of the above Propositions 2 and 3 have a symmetrical, equivalent formulation involving g.

3. Extended-order algebras

Definition 5. A *weak extended-order algebra*, shortly w-eo algebra, is a triplet $(L, \to, \top)$, where L is a non-empty set, $\to$: $L \times L \to L$ is a binary operation on L, $\top$ is a distinguished element of L such that for all $a, b, c \in L$ the following conditions are satisfied:

(o_1) $a \to \top = \top$ [upper bound condition]
(o_2) $a \to a = \top$ [reflexivity condition]
(o_3) if $a \to b = \top$ and $b \to a = \top$, then $a = b$ [antisymmetry condition]
(o_4) if $a \to b = \top$ and $b \to c = \top$, then $a \to c = \top$ [weak transitivity condition].

Proposition 6. *For any w-eo algebra $(L, \to, \top)$ the relation naturally associated to the operation $\to$ defined, for all $a, b \in L$, by*

$$a \leqslant b \quad \text{if and only if} \quad a \to b = \top$$

is an order relation in L. Moreover $\top$ is the top element in $(L, \leqslant)$. This order relation is called the natural ordering in L.

Proof. Reflexivity of $\leqslant$ follows from (o_2). Antisymmetry follows from (o_3). Transitivity follows from (o_4). (o_1) says that $\top$ is the greatest element of L. □

Conversely, the following statement can be easily proved.

Proposition 7. *If $(L, \leqslant)$ is a partially ordered set with a top element $\top$ and $\to: L \times L \to L$ extends $\leqslant$, then $(L, \to, \top)$ is a w-eo algebra.*

We note that this kind of structure has already been considered in a more general situation: in fact d-algebras considered in [30] are defined by the conditions $(o_1'): \bot \to a = \bot$, $\forall a \in L$, (o_2), (o_3) and need not be weak transitive, so w-eo algebras are the dual version of d-transitive d-algebras.

As already remarked in the introduction, w-eo algebras are exactly the implicative algebras considered in [32]; implication algebras and positive implication algebras (see [1,13,32]), BCK and pseudo-BCK algebras (see [26–28]) in their dual version, which we shall refer to in any case, and residuated lattices (see the discussion at the end of Section 5) are well known examples of w-eo algebras; each of these structures include or naturally determine a unique ordering in the underlying set.

Conversely, for every poset $(L, \leqslant, \top)$ with a greatest element there exist w-eo algebras $(L, \to, \top)$ whose natural ordering is $\leqslant$ (the operation $\to$ is uniquely determined if and only if $|L| \leqslant 2$): some but not all of these are either BCK or Hilbert algebras or residuated lattices, as the following easy examples show.

Example 8. Let $(L, \leqslant)$ be any poset with a greatest element $\top$ and define

$$a \to b = \begin{cases} \top & \text{if } a \leqslant b, \\ b & \text{if } a \nleqslant b. \end{cases}$$

$(L, \to, \top)$ is a well-known example of a Hilbert algebra and a BCK algebra and it is a w-eo algebra, of course. However it need not be a residuated lattice, even in case when $(L, \leqslant)$ is a (possibly complete) lattice, since the related product, which should be the meet operation, should be distributive over existing sups.

Example 9. Let $(L, \leqslant)$ be a poset with at least two elements and a greatest one $\top$. Fix any element $k \neq \top$ and for any $a, b \in L$ define

$$a \to b = \begin{cases} \top & \text{if } a \leqslant b, \\ k & \text{if } a \nleqslant b. \end{cases}$$

Obviously $(L, \to, \top)$ is a w-eo algebra, for every k, but it may not be either a Hilbert algebra or a BCK algebra. In fact, if $|L| > 2$, it is easy to disprove the equality $\top \to a = a$ for all $a \in L$, a necessary condition for Hilbert and BCK algebras.

In a w-eo algebra $(L, \to, \top)$, the axiom (o_4) says that the natural ordering $\leqslant$ on L satisfies transitivity. Now we consider two stronger axioms in L:

(o_5) If $a \to b = \top$, then $(c \to a) \to (c \to b) = \top$ [weak isotonic condition (in the second variable)]
(o_5') If $a \to b = \top$, then $(b \to c) \to (a \to c) = \top$ [weak antitonic condition (in the first variable)].

(o_5) says that $\to$ is isotonic for $\leqslant$ in the second argument, while (o_5') says that $\to$ is antitonic for $\leqslant$ in the first argument. Each of these axioms is a strengthening of (o_4) in a w-eo algebra. More precisely the following is clearly true.

Proposition 10. *If* $(L, \to, \top)$ *satisfies* (o_1) *and* (o_3)*, then*

- $\top \to x = \top \Leftrightarrow x = \top$,
- $(o_5) \Rightarrow (o_4)$ *and* $(o_5') \Rightarrow (o_4)$.

We also note that in a w-eo algebra (o_5) and (o_5') are independent of each other and that (o_4) does not imply none of them, even if (o_1), (o_2), (o_3) are also assumed. In fact the following examples show that there are w-eo algebras that do not satisfy (o_5) or (o_5') or both.

Example 11. Let $L = \{a, b, c, \top\}$ be the ordered set with top element $\top$ and $a \leqslant b$. Let the binary operation $\to$ be defined as follows:

$\to$	a	b	c	$\top$
a	$\top$	$\top$	b	$\top$
b	c	$\top$	c	$\top$
c	b	a	$\top$	$\top$
$\top$	b	b	c	$\top$

Then $(L, \to, \top)$ is a w-eo algebra that does not satisfy either (o_5') (since $a \to b = \top$ and $(b \to c) \to (a \to c) = c \to b = a \neq \top$) or (o_5) (since $a \to b = \top$ and $(c \to a) \to (c \to b) = b \to a = c \neq \top$).

Example 12. Let $L = \{a, b, c, d, \top\}$ be the ordered set with top element $\top$ and $a \leqslant b$, $c \leqslant d$. Let the binary operation $\to$ be defined as follows:

$\to$	a	b	c	d	$\top$
a	$\top$	$\top$	b	a	$\top$
b	b	$\top$	b	a	$\top$
c	b	b	$\top$	$\top$	$\top$
d	b	b	b	$\top$	$\top$
$\top$	b	b	b	a	$\top$

One can check that $(L, \to, \top)$ is a w-eo algebra that satisfies (o_5') but it does not satisfy (o_5). In fact, $c \leqslant d$ while $(\top \to c) \to (\top \to d) = b \to a = b \neq \top$.

Example 13. Let $L = \{a, b, c, d, \top\}$ be the ordered set with top element $\top$ and $a \leqslant b$, $c \leqslant d$. Let the binary operation $\to$ be defined as follows:

$\to$	a	b	c	d	$\top$
a	$\top$	$\top$	a	b	$\top$
b	a	$\top$	c	c	$\top$
c	a	a	$\top$	$\top$	$\top$
d	a	a	a	$\top$	$\top$
$\top$	a	a	c	c	$\top$

$(L, \to, \top)$ is a w-eo algebra that satisfies (o_5) but it does not satisfy (o_5'). In fact, $a \leqslant b$ while $(b \to d) \to (a \to d) = c \to b = a \neq \top$.

The above examples justify the following definitions.

Definition 14. The triplet $(L, \to, \top)$ is a *right w-eo algebra* if and only if it satisfies the axioms (o_1), (o_2), (o_3) and (o_5).

The triplet $(L, \to, \top)$ is a *left w-eo algebra* if and only if it satisfies the axioms (o_1), (o_2), (o_3) and (o_5').

Of course, it follows from Proposition 10 that both right w-eo algebras and left w-eo algebras are w-eo algebras.

Definition 15. $(L, \to, \top)$ is an *extended-order algebra*, shortly eo algebra, if it is a right and a left w-eo algebra.

Definition 16. $(L, \to, \top)$ is a *complete* (*weak*) *extended-order algebra*, shortly (w-)ceo algebra if it is a (weak) extended-order algebra and L is a complete lattice in the natural ordering.

Definition 17. The triplet $(L, \to, \top)$ is a *distributive extended-order algebra*, shortly deo algebra, if it is a w-eo algebra and satisfies the condition

(d) for any $A, A', B, B' \subseteq L$, if $Ub(A) = Ub(A')$ and $Lb(B) = Lb(B')$ then $Lb(A \to B) = Lb(A' \to B')$ [distributivity condition]

where, of course, $A \to B = \{a \to b \mid a \in A, b \in B\}$ and $A' \to B' = \{a' \to b' \mid a' \in A', b' \in B'\}$.

It is clear that one can also consider complete distributive extended-order algebras (cdeo algebras). The proposition below shows why we do not speak of distributive weak extended-order algebras.

Proposition 18. *Every distributive extended-order algebra* $(L, \to, \top)$ *is an extended-order algebra.*

Proof. To prove that (d) implies (o_5) and (o_5'), let $a, b \in L$ be such that $a \to b = \top$ and consider the set $\{a, b\}$. Then $\vee\{a, b\} = b$, $Lb(\{a, b\}) = Lb(\{a\})$ and $Ub(\{a, b\}) = Ub(\{b\})$. Now, by (d), with $A = \{a, b\}$, $A' = \{b\}$ and $B = B' = \{c\}$: $Lb(\{a \to c, b \to c\}) = Lb(\{a, b\} \to \{c\}) = Lb(\{b \to c\})$. In particular, $b \to c \in Lb(\{a \to c, b \to c\})$, which implies $b \to c \leqslant a \to c$, i.e. (o_5'). Furthermore, applying (d) to $B = \{a, b\}$, $B' = \{a\}$ and $A = A' = \{c\}$ one gets:

$Lb(\{c \to a, c \to b\}) = Lb(\{c\} \to \{a, b\}) = Lb(\{c \to a\})$. So $c \to a \in Lb(\{c \to a, c \to b\})$, then $c \to a \leqslant c \to b$, i.e. (o_5). $\square$

Remark 19. The axiom (d) is expressed by using typical notation in posets; however the operators Ub and Lb can also refer to a more general binary relation r than an ordering in a set L (they could be denoted Lb_r, Ub_r in this case); so the axiom (d) makes sense in more general structures (for instance d-algebras of [30] whose natural associated relation is not an ordering) than w-eo algebras. Indeed the following example shows that d-algebras (in their dual form where (o_1') is replaced by (o_1)), that satisfy the axiom (d) need not be w-eo algebras. In other terms if (o_1), (o_2) and (o_3) are assumed to hold, then (o_5) (or (o_5') as well) implies (o_4), (d) and (o_4) imply (o_5) and (o_5') but (d) need not imply (o_4).

Example 20. Consider the set $L = \{1, 2, 3, 4\}$ and for all $x, y \in L$ define $x \to y = 4$ whenever $0 \leqslant y - x \leqslant 1$ and $x \to y = y$ otherwise.

Clearly $(L, \to, 4)$ is a d-algebra with a non-transitive natural associated relation r, so it is not a w-eo algebra. Nevertheless one can check that, with obvious notation, $\forall A, A', B, B' \subseteq L$ the following implication holds $Ub_r(A) = Ub_r(A'), Lb_r(B) = Lb_r(B') \Rightarrow Lb_r(A \to B) = Lb_r(A' \to B')$.

Proposition 21. *Let $(L, \to, \top)$ be a w-ceo algebra. Then the distributivity condition (d) is equivalent to the following*

(d') *for any $A, B \subseteq L$, $\bigvee A \to \bigwedge B = \bigwedge(A \to B)$.*

Proof. Assume (d) and let $A, B \subseteq L$. It follows from $Ub(\bigvee A) = Ub(A)$ and $Lb(\bigwedge B) = Lb(B)$ that $Lb(\bigvee A \to \bigwedge B) = Lb(A \to B)$. But $Lb(\bigwedge(A \to B)) = Lb(A \to B)$, too. Hence $\bigvee A \to \bigwedge B = \bigwedge\{\bigvee A \to \bigwedge B\} = \bigwedge(A \to B)$.

Conversely, assume (d') and let $A, A', B, B' \subseteq L$ be such that $Ub(A) = Ub(A')$ and $Lb(B) = Lb(B')$. Then $\bigvee A = \bigvee A'$ and $\bigwedge B = \bigwedge B'$, and consequently $Lb(A \to B) = Lb(\{\bigwedge(A \to B)\}) = Lb(\{\bigvee A \to \bigwedge B\}) = Lb(\{\bigvee A' \to \bigwedge B'\}) = Lb(A' \to B')$. $\square$

Similarly, the following result can be proved:

Proposition 22. *If $(L, \to, \top)$ is a deo algebra and $A, B \subseteq L$, then*

$$\bigvee A \neq \emptyset, \bigwedge B \neq \emptyset \Rightarrow \bigvee A \to \bigwedge B = \bigwedge(A \to B).$$

4. MacNeille completion of extended-order algebras

In this section we shall adopt a slightly different description of the MacNeille completion of a poset with a top element [29] and apply such a construction to embed (some kind of) extended-order algebras into complete extensions. We first recall some basic well known definitions and statements to introduce and clarify the notation we adopt. A different approach to the MacNeille completion can be found in [7], too.

Let $(L, \leqslant, \top)$ be a poset with greatest element $\top$. Define in $\mathcal{P}(L)$ the binary relation $A \approx A'$ if $Lb(A) = Lb(A')$, for any $A, A' \subseteq L$.

Trivially, $\approx$ is an equivalence relation in $\mathcal{P}(L)$. Let K be the quotient set, i.e. $K = \mathcal{P}(L)/\approx$, and denote by $[A]$ the equivalence class of $A \subseteq L$; $[a]$ stands for $[\{a\}]$, if $a \in L$.

Lemma 23.

1) *If $\alpha = [A_i]$ for all $i \in I$, then $\alpha = [\bigcup_{i \in I} A_i]$.*
2) *If $\alpha = [A]$, then $Ub(Lb(A)) = \cup\{X \mid [X] = \alpha\}$ is the largest representative of α, i.e. $[A] = [Ub(Lb(A))]$ and if $[A] = [A']$ then $A' \subseteq Ub(Lb(A))$.*
3) *If $a = \bigwedge A \in L$ exists, then $[A] = [a] = [\uparrow a]$.*

4) *If* $[a]=[a']$, *then* $a=a'$.
5) *For every* $\alpha\in K$ *there exists* $A\neq\emptyset$: $\alpha=[A]$.

Proof. 1): By assumption, for all $i,j\in I$, $Lb(A_i)=Lb(A_j)$ and then $Lb(\bigcup_{i\in I}A_i)=\bigcap_{i\in I}Lb(A_i)=Lb(A_j)$, for all $j\in I$. Thus, $\alpha=[\bigcup_{i\in I}A_i]$.

2): Since $Lb(Ub(Lb(A)))=Lb(A)$ then $[A]=[Ub(Lb(A))]$. Now it follows from $[A']=[A]$ that $A'\subseteq Ub(Lb(A'))=Ub(Lb(A))$.

3): Clearly, $Lb(A)=Lb(\bigwedge A)$ and $Ub(Lb(a))=\uparrow a$, hence the equalities follow trivially.

4): If $[a]=[a']$, then by 3) $[\uparrow a]=[\uparrow a']$. Then $Lb(\uparrow a)=Lb(\uparrow a')$, which implies that $a\in Lb(\uparrow a')$ and $a'\in Lb(\uparrow a)$, i.e. $a\leqslant a'$ and $a'\leqslant a$. Thus $a=a'$.

5): It is clear since $[\emptyset]=[\top]$. □

Consider the binary relation in K defined for any $\alpha,\beta\in K$, $\alpha=[A]$, $\beta=[B]$, by

$$\alpha\leqslant_K\beta\quad\text{if and only if}\quad Lb(A)\subseteq Lb(B).$$

Clearly this condition is equivalent to the inclusion $Ub(Lb(B))\subseteq Ub(Lb(A))$ and the relation $\leqslant_K$ is an order relation in K.

We also note that $[A]\leqslant_K[B]$ if $B\subseteq A$.

If $\varphi:L\to K$ is defined by

$$a\mapsto\varphi(a)=[a]$$

then φ is an embedding. In fact φ is injective and if $a,b\in L$ then

$$a\leqslant b\Leftrightarrow Lb(a)\subseteq Lb(b)\Leftrightarrow\varphi(a)\leqslant_K\varphi(b).$$

$(K,\leqslant_K)$ is a complete lattice with greatest element $\top_K=[\top]$, least element $\bot_K=[L]$ and moreover, for any $S\subseteq K$,

$$\bigvee\nolimits_K S=\bigl[\cap\bigl\{Ub\bigl(Lb(A)\bigr)\mid[A]\in S\bigr\}\bigr]\quad\text{and}\quad\bigwedge\nolimits_K S=\bigl[Ub\bigl(\cap\bigl\{Lb(A)\mid[A]\in S\bigr\}\bigr)\bigr].$$

In fact, as already remarked, the complete lattice $(K,\leqslant_K)$ above described is nothing but the MacNeille completion of $(L,\leqslant)$ defined in [29] as the set of cuts $M=\{[A,A']\mid A=Lb(A'),\,A'=Ub(A)\}$ in L; our approach, indeed, does not consider a lower bound in L.

More explicitly the correspondences

$$[B]\in K\mapsto\bigl[Lb(B),Ub\bigl(Lb(B)\bigr)\bigr]\in M$$

and

$$[A,A']\in M\mapsto[A']\in K$$

determine isomorphisms of complete lattices which are inverses of each other, mainly in case when L is bounded.

Now we list a few statements that follow easily from the above definitions and from Lemma 23 and will be useful to consider.

Remark 24.

1) For $S\subseteq K$ and $\mathcal{S}=\{A\subseteq L\mid[A]\in S\}$ let $\sigma:\mathcal{S}\to S$ be the surjective correspondence mapping $A\in\mathcal{S}$ to $[A]\in S$. If $\mathcal{S}'$ is any subset of $\mathcal{S}$, such that the restriction of σ to $\mathcal{S}'$ is a surjection onto S, then $Lb(\mathcal{S}')=\{Lb(A')\mid A'\in\mathcal{S}'\}=\{Lb(A)\mid A\in\mathcal{S}\}=Lb(\mathcal{S})$, $Lb(\cup\mathcal{S}')=Lb(\cup\mathcal{S})$, hence $[\cup\mathcal{S}']=[\cup\mathcal{S}]$.
Sometimes it is useful to mean that $\{A\mid[A]\in S\}$ or $\{Lb(A)\mid[A]\in S\}$ denote, more simply, $\mathcal{S}'$ or $Lb(\mathcal{S}')$ with $\mathcal{S}'$ any subset of $\mathcal{S}$ given by choosing one representative of each element of S: this notation already occurred in the above description of sups and infs in K and it applies to similar situations, as below.
2) If $B\subseteq K$, then it follows from the recalled properties of the Galois connection $[Lb,Ub]$ that

$$\bigwedge\nolimits_K B=\bigl[Ub\bigl(Lb\bigl(\cup\bigl\{A\mid[A]\in B\bigr\}\bigr)\bigr)\bigr]=\bigl[\cup\bigl\{A\mid[A]\in B\bigr\}\bigr]$$

and

$$\bigvee_K B = \big[Ub\big(\cup\big\{Lb(A) \mid [A] \in B\big\}\big)\big].$$

3) If $A \subseteq L$, then $\bigwedge_K \varphi(A) = \bigwedge_K \{[a] \mid a \in A\} = [Ub(Lb(A))] = [A]$. Hence $\bigwedge_K A' = \bigwedge_K A$ if and only if $[A'] = [A]$.

Similarly, if $B \subseteq L$, $\bigvee_K \varphi(B) = [\cap\{Ub(Lb(\{b\})) \mid b \in B\}] = [\cap\{Ub(\{b\}) \mid b \in B\}] = [Ub(\cup\{\{b\} \mid b \in B\})] = [Ub(B)]$.

Definition 25. Let $(L, \to, \top)$ be a w-eo algebra and let $(K, \leqslant_K)$ be the MacNeille completion of $(L, \leqslant, \top)$. For any $\alpha = [A], \beta = [B] \in K$, define

$$\alpha \to_K \beta = \big[Lb(A) \to Ub\big(Lb(B)\big)\big].$$

Proposition 26. *With the above notation the following hold*:

(1) $\to_K$ *is an extension of* $\leqslant_K$, *i.e. for all* $\alpha, \beta \in K$

$$\alpha \to_K \beta = \top_K \Leftrightarrow \alpha \leqslant_K \beta.$$

(2) $(K, \to_K, \top_K)$ *is a complete w-eo algebra.*

Proof. Let $\alpha = [A], \beta = [B] \in K$ and let $A \neq \emptyset \neq B$. Then

$$\alpha \to_K \beta = \top_K \Leftrightarrow Lb(A) \to Ub\big(Lb(B)\big) = \{\top\} \Leftrightarrow Lb(A) \subseteq Lb\big(Ub\big(Lb(B)\big)\big) = Lb(B) \Leftrightarrow \alpha \leqslant_K \beta.$$

So, (1) is true and then (2) follows easily by Proposition 7. □

Lemma 27. *With the above notation the following results hold*:

(1) *If* $(L, \to, \top)$ *is an eo algebra, then* $\to_K$ *is an extension of* $\to$, *i.e.*

$$[a] \to_K [b] = [a \to b], \quad \textit{for all } a, b \in L.$$

(2) *If* $(L, \to, \top)$ *is a distributive eo algebra, then for all* $\alpha = [A], \beta = [B] \in K$ *one has* $\alpha \to_K \beta = [Lb(A) \to B]$.

Proof. (1) Clearly $[a] \to_K [b] = [\downarrow a \to \uparrow b] \leqslant_K [a \to b]$. Furthermore, it follows from the assumption that $\downarrow a \to \uparrow b \subseteq Ub(a \to b)$, thence $Lb(a \to b) = Lb(Ub(a \to b)) \subseteq Lb(\downarrow a \to \uparrow b)$, which proves the converse inequality.

(2) It follows by the assumption and by the equality $Lb(Ub(Lb(B))) = Lb(B)$ that $Lb(Lb(A) \to Ub(Lb(B))) = Lb(Lb(A) \to B)$ which proves the stated equality. □

Remark 28. The assumption that $(L, \to, \top)$ be a (left or right) w-eo algebra would not be enough to prove that $\to_K$ is an extension of $\to$. To show this we describe the MacNeille completion of the left w-eo algebra L of Example 12. The elements of the completion are:

- $\bot_K = [L]$;
- $\alpha = [a]$;
- $\beta = [b]$;
- $\gamma = [c]$;
- $\delta = [d]$;
- $\top_K = [\top]$.

The extended order $\to_K$ is described in the following table:

$\to_K$	$\bot_K$	α	β	γ	δ	$\top_K$
$\bot_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$
α	α	$\top_K$	$\top_K$	α	α	$\top_K$
β	α	β	$\top_K$	α	α	$\top_K$
γ	β	β	β	$\top_K$	$\top_K$	$\top_K$
δ	β	β	β	β	$\top_K$	$\top_K$
$\top_K$	α	β	β	α	α	$\top_K$

One can see that $\beta \to_K \gamma = \alpha$, while $b \to c = b$.

Similarly, in the completion of the w-eo algebra of Example 13, one can see that $[b] \to_K [c] = \bot_K$, while $b \to c = c$.

Proposition 29. *Let $(L, \to, \top)$ be an eo algebra. Then, assuming the above notation, the following hold*:

(1) *$(K, \to_K, \top_K)$ is a complete eo algebra and the map $\varphi : L \to K$ defined by $\varphi(a) = [a]$ is an embedding*;
(2) *if $(L, \to, \top)$ is distributive, then $(K, \to_K, \top_K)$ is distributive.*

Proof. (1): We only need to prove that (o_5) and (o_5') hold, since the remaining part of the statement follows trivially by the above discussion and by Lemma 27. Let $[A] \leqslant_K [B]$, $A, B \subseteq L$. Then $Lb(A) \subseteq Lb(B)$ and for every $[C] \in K$ one has $[Lb(C) \to Ub(Lb(A))] \leqslant_K [Lb(C) \to Ub(Lb(B))]$ and $[Lb(B) \to Ub(Lb(C))] \leqslant_K [Lb(A) \to Ub(Lb(C))]$, that is $[C] \to_K [A] \leqslant_K [C] \to_K [B]$ and $[B] \to_K [C] \leqslant_K [A] \to_K [C]$.

(2): For every $S, T \subseteq K$ we now prove that $\bigvee_K S \to_K \bigwedge_K T = \bigwedge_K (S \to_K T)$. Assume first $S = \{\alpha\}$ and $\alpha = [A] \in K$, so $\bigvee_K S = \alpha$. It follows from $\bigwedge_K T = [\cup\{B \mid [B] \in T\}]$ and from Lemma 27 that $\alpha \to_K \bigwedge_K T = [Lb(A) \to \cup\{B \mid [B] \in T\}]$. On the other hand $\alpha \to_K T = \{\alpha \to_K [B] \mid [B] \in T\} = \{[Lb(A) \to B] \mid [B] \in T\}$. Therefore $\bigwedge_K (\alpha \to_K T) = [\cup\{Lb(A) \to B \mid [B] \in T\}]$. We get the awaited result since clearly $\cup\{Lb(A) \to B \mid [B] \in T\} = Lb(A) \to \cup\{B \mid [B] \in T\}$.

Assume now that $T = \{\beta\}$, $\beta = [B]$. Since $\bigvee_K S = [Ub(\cup\{Lb(A) \mid [A] \in S\})]$, then by Lemma 27 $\bigvee_K S \to_K \beta = [Lb(Ub(\cup\{Lb(A) \mid [A] \in S\})) \to B]$. On the other hand $S \to_K \beta = \{[A] \to_K \beta \mid [A] \in S\} = \{[Lb(A) \to B] \mid [A] \in S\}$. Thence $\bigwedge_K (S \to_K \beta) = [\cup\{Lb(A) \to B \mid [A] \in S\}]$. Since $Ub(Lb(Ub(\cup\{Lb(A) \mid [A] \in S\}))) = Ub(\cup\{Lb(A) \mid [A] \in S\})$, it follows by the distributivity assumption on L that $\bigvee_K S \to_K \beta = [Lb(Ub(\cup\{Lb(A) \mid [A] \in S\})) \to B] = [\cup\{Lb(A) \mid [A] \in S\} \to B] = [\cup\{Lb(A) \to B \mid [A] \in S\}] = \bigwedge_K (S \to_K \beta)$.

Now the proof in the case of arbitrary $S, T \subseteq K$ follows easily since $\bigvee_K S \to_K \bigwedge_K T = \bigwedge_K \{\bigvee_K S \to_K \beta \mid \beta \in T\} = \bigwedge_K \{\alpha \to_K \beta \mid \alpha \in S, \beta \in T\} = \bigwedge_K (S \to_K T)$. $\square$

We call $(K, \to_K, \top_K)$ the *MacNeille completion of the w-eo algebra* $(L, \to, \top)$.

Remark 30.

(1) We emphasize that the results proved in Proposition 29 allow to consider the completeness condition a not too strong requirement for eo algebras and to realize that the distributivity condition of eo algebras is not invalidated by the additional assumption of the completeness condition.
(2) We note that in any complete distributive extended-order algebra (shortly cdeo algebra) $(L, \to, \top)$ the distributivity condition (d') can be expressed equivalently by means of the following two conditions:
  (d_1) $a \in L, B \subseteq L \Rightarrow a \to \bigwedge B = \bigwedge(a \to B)$,
  (d_2) $A \subseteq L, b \in L \Rightarrow \bigvee A \to b = \bigwedge(A \to b)$.

5. The adjoint product

The structure of cdeo algebra, which can be obtained, as it was shown in the previous sections, by suitable conditions on the extension $\to$ of the ordering of any poset with a top element seems to be a good candidate to be taken as a quite general range for lattice-valued mathematics: it has the main needed tools to do that, though it is much weaker than a (generalized) complete residuated lattice, mostly used up to now.

Now let $(L, \to, \top)$ be a cdeo algebra. Then, for all $a, b \in L$ we define:

$$a \otimes b = \bigwedge\{x \in L \mid b \leqslant a \to x\}.$$

Thus, by Theorem 1, $\otimes$ and $\to$ form an adjoint pair, i.e. for all $x, y, z \in L$:

$$x \otimes y \leqslant z \Leftrightarrow y \leqslant x \to z.$$

Such a product, that we call the *adjoint product* of the cdeo algebra, is very important but we shall not include it explicitly in our notation for the structure of cdeo algebras that will remain as follows: $(L, \to \top)$. As a motivation for such a choice we note that $\otimes$ is completely and univocally determined by $\to$; moreover we shall see that further important conditions, such as associativity and commutativity of the product can be completely characterized in terms of the extended order $\to$ only.

In this section we shall state and prove most basic properties that are well known in complete residuated lattices: we list them in different steps to make it clear which of those hold in any cdeo algebra and which of those require further assumptions.

Proposition 31. *Let $(L, \to, \top)$ be a cdeo algebra. Then, for all $a, b, c \in L$ and for all $A, B \subseteq L$, the following properties hold*:

(1) $a \otimes b \leqslant a$,
(2) $a \otimes \top = a$,
(3) $a \otimes \bot = \bot$,
(4) $\bot \otimes a = \bot$,
(5) $\top \otimes a \leqslant a$ *if and only if* $\top \to a \geqslant a$,
(5′) $\top \otimes a \geqslant a$ *if and only if* $(\forall x \in L : \top \to x \geqslant a \Rightarrow x \geqslant a)$,
(6) $a \otimes (\bigvee B) = \bigvee(a \otimes B)$,
(6′) *if* $b \leqslant c$, *then* $a \otimes b \leqslant a \otimes c$,
(7) $(\bigvee A) \otimes b \geqslant \bigvee(A \otimes b)$,
(7′) *if* $a \leqslant b$, *then* $a \otimes c \leqslant b \otimes c$,
(8) $a \otimes (a \to b) \leqslant b \leqslant a \to (a \otimes b)$.

Proof. Let $a, b, c \in L$ and $A, B \subseteq L$. Then

(1): From (o_2), it follows that $b \leqslant a \to a$, hence $a \otimes b \leqslant a$.

(2): $a \otimes \top = \bigwedge\{x \in L \mid \top \leqslant a \to x\} = a$, since $\top \leqslant a \to a$ and $\top \leqslant a \to x$ implies $a \leqslant x$.

(3): $a \otimes \bot = \bigwedge\{x \mid \bot \leqslant a \to x\} = \bigwedge L = \bot$.

(4): It follows from (1).

(5): It follows from the adjunction condition.

(5′): Let $\top \otimes a \geqslant a$ and let $x \in L$ be such that $\top \to x \geqslant a$. Then by the adjunction condition $\top \otimes a \leqslant x$, so $x \geqslant a$. Conversely, let $a \in L$ and let $x \geqslant a$ whenever $\top \to x \geqslant a$. Then $\top \otimes a \geqslant a$ since clearly $\top \to (\top \otimes a) \geqslant a$.

(6): It follows from Theorem 1.

(6′): From (6) and from the assumption it follows that $(a \otimes b) \vee (a \otimes c) = a \otimes (b \vee c) = a \otimes c$.

(7): $(\bigvee A) \otimes b = \bigwedge\{x \mid b \leqslant (\bigvee A) \to x\} = \bigwedge\{x \mid b \leqslant \bigwedge(A \to x)\} = \bigwedge(\cap\{\{t \mid b \leqslant a \to t\} \mid a \in A\}) \geqslant \bigvee\{\bigwedge\{t \mid b \leqslant a \to t\} \mid a \in A\} = \bigvee(A \otimes b)$.

(7′): From (7) and from the assumption it follows that $(a \otimes c) \vee (b \otimes c) \leqslant (a \vee b) \otimes c = b \otimes c$. But clearly: $(a \otimes c) \vee (b \otimes c) \geqslant b \otimes c$, and thence (7′) follows.

(8): In fact, the first inequality is equivalent to $a \to b \leqslant a \to b$; the second one is equivalent to $a \otimes b \leqslant a \otimes b$. $\square$

Remark 32. Putting together (5) and (5′) of the above proposition, for all $a \in L$, we get:

(5″) $\top \otimes a = a$ iff $(\top \to x \geqslant a \Leftrightarrow x \geqslant a)$.

Definition 33. An eo algebra $(L, \to, \top)$ is *commutative* if and only if it satisfies the weak exchange condition:

(c) $a \to (b \to c) = \top \Leftrightarrow b \to (a \to c) = \top$,

for all $a, b, c \in L$.

Proposition 34. *The MacNeille completion of a commutative deo algebra is still commutative.*

Proof. Let $(L, \to, \top)$ be a commutative deo algebra and let $(K, \to_K, \top_K)$ be its MacNeille completion. Let $A, B, C \subseteq L$ be non-empty representatives of $\alpha, \beta, \gamma \in K$, and assume α, β, γ to be such that $\alpha \to_K (\beta \to_K \gamma) = \top_K$. Then $Lb(A) \to (Lb(B) \to C) \subseteq \{\top\}$. If $Lb(A) \neq \emptyset$ and $Lb(B) \neq \emptyset$, then $a' \to (b' \to c) = \top$, for all $a' \in Lb(A)$, $b' \in Lb(B)$ and $c \in C$. From (c) it follows that $b' \to (a' \to c) = \top$, for all $a' \in Lb(A)$, $b' \in Lb(B)$ and $c \in C$. Thence, $Lb(B) \to (Lb(A) \to C) \subseteq \{\top\}$, so that $\beta \to_K (\alpha \to_K \gamma) = \top_K$. Clearly, if either $Lb(A) = \emptyset$ or $Lb(B) = \emptyset$ the assertion is trivially proved, since $[\emptyset] = \top_K$. □

Proposition 35. *A cdeo algebra $(L, \to, \top)$ is commutative if and only if its adjoint product $\otimes$ is commutative.*

Proof. The statement follows trivially from the definition and from the following equivalences:

$$\begin{aligned} a \otimes b \leqslant b \otimes a &\Leftrightarrow b \leqslant a \to (b \otimes a) \\ &\Leftrightarrow b \leqslant a \to \bigwedge\{x \mid a \leqslant b \to x\} \\ &\Leftrightarrow b \leqslant \bigwedge_x \{a \to x \mid a \leqslant b \to x\} \\ &\Leftrightarrow b \leqslant a \to x \quad \text{ if } a \leqslant b \to x \\ &\Leftrightarrow b \to (a \to x) = \top \quad \text{ if } a \to (b \to x) = \top. \qquad \square \end{aligned}$$

Definition 36. An eo algebra $(L, \to, \top)$ is *associative* if and only if it satisfies the following condition:

(a) $Lb(Lb(A) \to (Lb(B) \to C)) = Lb(Lb(\{x \mid Lb(A) \subseteq Lb(Lb(B) \to x)\}) \to C)$

for all $A, B, C \subseteq L$.

Lemma 37. *If $(L, \to, \top)$ is a cdeo algebra, then the condition (a) can be formulated in the following equivalent way*:

(a') $a \to (b \to c) = (\bigwedge\{x \mid a \to (b \to x) = \top\}) \to c$,

for all $a, b, c \in L$.

Proof. By the completeness assumption the equality in (a) becomes

$$\bigwedge\big(Lb(A) \to \big(Lb(B) \to C\big)\big) = \bigwedge\big(\big(Lb\big\{x \big| \bigwedge A \leqslant \bigwedge\big(Lb(B) \to x\big)\big\}\big) \to C\big)$$

hence, by also using distributivity, (a) is equivalent to

$$\bigwedge A \to \big(\bigwedge B \to \bigwedge C\big) = \big(\bigwedge\big\{x \big| \bigwedge A \leqslant \bigwedge B \to x\big\}\big) \to \bigwedge C, \quad \forall A, B, C \subseteq L.$$

Now, thanks to the above equality, (a') follows from (a) taking $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, while (a) follows from (a') taking $a = \bigwedge A$, $b = \bigwedge B$, $c = \bigwedge C$. □

Proposition 38. *Let $(L, \to, \top)$ be a deo algebra and let $(K, \to_K, \top_K)$ be its MacNeille completion, with the above notation. If L is associative, then K is associative, too.*

Proof. Let $\alpha = [A]$, $\beta = [B]$, $\gamma = [C]$ be in K, with $A, B, C \subseteq L \setminus \{\emptyset\}$. Denote

$$X_{\alpha\beta} = \cup\big\{X \mid Lb(A) \subseteq Lb\big(Lb(B) \to X\big)\big\}.$$

From Remark 24 it follows that

$$\bigwedge\nolimits_K \{[X] \mid \alpha \to_K (\beta \to_K [X]) = \top_K\} = [X_{\alpha\beta}];$$

moreover

$$Lb(A) \subseteq Lb(Lb(B) \to X_{\alpha\beta});$$

in fact,

$$\begin{aligned} Lb(Lb(B) \to X_{\alpha\beta}) &= Lb(Lb(B) \to \cup\{X \mid Lb(A) \subseteq Lb(Lb(B) \to X)\}) \\ &= Lb(\cup\{Lb(B) \to X \mid Lb(Lb(B) \to X) \supseteq Lb(A)\}) \\ &= \cap\{Lb(Lb(B) \to X) \mid Lb(Lb(B) \to X) \supseteq Lb(A)\} \\ &\supseteq Lb(A). \end{aligned}$$

By Lemma 37 we have to prove now that

$$\alpha \to_K (\beta \to_K \gamma) = \left(\bigwedge\nolimits_K \{[X] \in K \mid \alpha \to_K (\beta \to_K [X]) = \top_K\}\right) \to_K \gamma,$$

which means

$$[Lb(A) \to (Lb(B) \to C)] = [Lb(X_{\alpha\beta}) \to C]$$

or equivalently

$$Lb(Lb(A) \to (Lb(B) \to C)) = Lb(Lb(X_{\alpha\beta}) \to C)$$

which follows by the assumption since the following equality

$$X_{\alpha\beta} = \{x \in L \mid Lb(A) \subseteq Lb(Lb(B) \to x)\}$$

holds; in fact, with the above notation one has:

$$\begin{aligned} x \in X_{\alpha\beta} &\Leftrightarrow \exists X, x \in X, Lb(A) \subseteq Lb(Lb(B) \to X) \\ &\Leftrightarrow Lb(A) \subseteq Lb(Lb(B) \to \{x\}). \qquad \square \end{aligned}$$

Remark 39. Let $(L, \to, \top)$ be a cdeo algebra. If $\otimes$ denotes the adjoint product, then the associativity condition (a) can be expressed in another equivalent way:

(a'') $a \to (b \to c) = (b \otimes a) \to c$,

for all $a, b, c \in L$.

Proposition 40. *Let $(L, \to, \top)$ be a cdeo algebra. Then its adjoint product $\otimes$ is associative if and only if L is associative.*

Proof. Suppose that L is associative. Then the inequality $(a \otimes b) \otimes c \leqslant a \otimes (b \otimes c)$ follows, by (a''), from the equivalences:

$$\begin{aligned} c \leqslant (a \otimes b) \to (a \otimes (b \otimes c)) &\Leftrightarrow c \leqslant (a \otimes b) \to \bigwedge\{x \mid b \otimes c \leqslant a \to x\} \\ &\Leftrightarrow c \leqslant (a \otimes b) \to \bigwedge\{x \mid c \leqslant b \to (a \to x)\} \\ &\Leftrightarrow c \leqslant \bigwedge_x \{(a \otimes b) \to x \mid c \leqslant (a \otimes b) \to x\}. \end{aligned}$$

To check the converse inequality, $a \otimes (b \otimes c) \leqslant (a \otimes b) \otimes c$, consider the following equivalences and use, once more, (a''):

$$b \otimes c \leqslant a \to ((a \otimes b) \otimes c) \Leftrightarrow c \leqslant b \to (a \to ((a \otimes b) \otimes c))$$

$$\Leftrightarrow c \leqslant b \to \Bigl(a \to \Bigl(\bigwedge\{x \mid c \leqslant (a \otimes b) \to x\}\Bigr)\Bigr)$$
$$\Leftrightarrow c \leqslant b \to \Bigl(\bigwedge_x \{a \to x \mid c \leqslant (a \otimes b) \to x\}\Bigr)$$
$$\Leftrightarrow c \leqslant \bigwedge_x \{b \to (a \to x) \mid c \leqslant b \to (a \to x)\}.$$

Assume now $\otimes$ to be associative. The inequality $a \to (b \to c) \leqslant (b \otimes a) \to c$ follows from Proposition 31(8), since $b \otimes (a \otimes (a \to (b \to c))) \leqslant b \otimes (b \to c) \leqslant c$. We get the converse inequality by Proposition 31(8) and by the following equivalences:

$$(b \otimes a) \to c \leqslant a \to (b \to c) \Leftrightarrow a \otimes \bigl((b \otimes a) \to c\bigr) \leqslant b \to c$$
$$\Leftrightarrow (b \otimes a) \otimes \bigl((b \otimes a) \to c\bigr) \leqslant c. \qquad \square$$

Remark 41. The attribute commutative and associative we have given to eo algebras of Definitions 33, 36 do not refer in the usual sense to the (unique) binary operation $\to$ of the eo algebra. Nevertheless we believe that this unusual terminology does not cause misunderstanding because of the unsymmetrical character of the operation $\to$, which cannot be expected to satisfy commutativity in non-trivial cases and, as remarked in [30] for d-algebras, provides the "most non-associative" algebras (in the usual sense).

We also note that our commutativity condition has nothing to do with the (pseudo) commutativity considered in BCK algebras or in Hilbert algebras (see [10,21]).

Proposition 42. *Let $(L, \to, \top)$ be a cdeo algebra and let $\otimes$ be its adjoint product.*

If L is commutative, then:

(1) $\top$ *is the neutral element for* $\otimes$;
(2) $a \otimes b \leqslant b \leqslant a \to b$;
(3) $a \otimes b \leqslant a \wedge b$;
(4) $a \leqslant b \to (a \otimes b)$;
(5) $a \leqslant (a \to b) \to b$;
(6) $\top \to a = a$;
(7) $(\bigvee A) \otimes b = \bigvee (A \otimes b)$.

If L is associative, then:

(8) $(a \to b) \otimes (b \to c) \leqslant a \to c$;
(8′) $(b \to c) \to ((a \to b) \to (a \to c)) = \top$.

If L is associative and commutative, then:

(9) $a \to (b \to c) = b \to (a \to c)$;
(10) $(a \to b) \to ((b \to c) \to (a \to c)) = \top$;
(11) $a \otimes (b \to c) \leqslant b \to (a \otimes c)$;

for all $a, b, c \in L$.

Proof. (1): It follows from Proposition 31(2) and from the commutativity property of $\otimes$.

(2): In fact, the first inequality follows from Proposition 31(1) and the commutativity property; the second inequality is equivalent to the first one.

(3): It follows from (2) and from Proposition 31(1).

(4): This inequality is equivalent to $b \otimes a \leqslant a \otimes b$.

(5): From the commutativity property of $\otimes$ and from Proposition 31(8) it follows that $(a \to b) \otimes a = a \otimes (a \to b) \leqslant b$, which is equivalent to the assertion.

(6): From (2), it follows that $\top \to a \geqslant a$ and from (5) the converse inequality follows.

(7): It follows from Proposition 31(5) and from the commutativity property of $\otimes$.

(8): It follows from the associativity property of $\otimes$ and from Proposition 31(6′) and (8). In fact it is true that

$$a \otimes \big((a \to b) \otimes (b \to c)\big) = \big(a \otimes (a \to b)\big) \otimes (b \to c) \leqslant b \otimes (b \to c) \leqslant c.$$

(8′): It is clearly equivalent to (8).

(9): It follows from (a'') and from the commutativity property of $\otimes$ that $a \to (b \to c) = (b \otimes a) \to c = (a \otimes b) \to c = b \to (a \to c)$, then the assertion holds.

(10): It follows from (8) and from the commutativity property of $\otimes$.

(11): In fact, by Proposition 31(8) and the assumption, the inequality $b \otimes (a \otimes (b \to c)) = a \otimes (b \otimes (b \to c)) \leqslant a \otimes c$ is true. $\square$

Remark 43. For any eo algebra $(L, \to, \top)$ all the properties of the operation $\to$ that are satisfied in its completion, still hold in L, thanks to the embedding described in Section 4. In particular, some important properties, that we only needed to assume in their weak version, can be extended to their strong version, thanks to the above results. In fact the following statements are clearly true.

Corollary 44. *Let $(L, \to, \top)$ be a deo algebra. Then*

(1) *the MacNeille completion $(K, \to_K, \top_K)$ is distributive* (*associative, commutative, respectively*) *if and only if $(L, \to, \top)$ is distributive* (*associative, commutative, respectively*);
(2) *if L is associative, then the* (*strong*) *isotonic condition*

$$(b \to c) \leqslant (a \to b) \to (a \to c)$$

holds;
(3) *if L is associative and commutative, then the* (*strong*) *exchange condition*

$$a \to (b \to c) = b \to (a \to c)$$

and the (*strong*) *antitonic condition*

$$(a \to b) \leqslant (b \to c) \to (a \to c)$$

hold.

Now, the above results allow to check easily the following statements, where $(M, \to, \top)$ is meant to be a cdeo algebra with top $\top$, bottom $\bot$ and adjoint product $\otimes$; further assumptions will be specified when needed.

- $(M, \otimes)$ is a *po-groupoid* (see [7]), with a zero element (its bottom $\bot$); moreover one can say, by partly extending Birkhoff's notation, that $(M, \otimes)$ is a *right integral right cl-groupoid*, which means that
  - M is a complete lattice and the product $\otimes$ is isotonic in both variables (i.e. $a \leqslant b \Rightarrow a \otimes c \leqslant b \otimes c$ and $c \otimes a \leqslant c \otimes b$) and it distributes on arbitrary sups on the right side (i.e. $a \otimes (\bigvee B) = \bigvee (a \otimes B)$);
  - M has a right unit 1 (i.e. $a \otimes 1 = a$); namely in our case $1 = \top$, hence
  - M is integral (i.e $a \leqslant \top$, for all $a \in M$);

  Moreover, every element $a \in M$ is right ideal (i.e. $a \otimes x \leqslant a$, for all $a, x \in M$).

In fact, these structures take sense from their origin, here described, and their possible applications: for instance, the result of Corollary 1, page 327 of [7], can be extended by saying

any right cl-groupoid with zero is right residuated.

- If $(M, \to, \top)$ is associative, then similarly one can speak of $(M, \otimes)$ as a *po-semigroup* with zero and a *right integral right cl-semigroup* whose elements are right ideal.

- Eventually, if $(M, \to, \top)$ is associative and commutative then $(M, \otimes)$ is a *cl-integral commutative monoid* with zero, whose elements are ideal.

In the last case, indeed, it is more informative to say that $(M, \vee, \wedge, \otimes, \to, \bot, \top)$ is a *complete residuated lattice*, a structure very frequently addressed in many-valued mathematics and logics and applications: t-norms, BL-algebras, MV-algebras (see [11,19,22]) and more specifically Heyting algebras and boolean algebras are included. More explicitly one can say that complete residuated lattices are the same things as associative commutative cdeo algebras.

The statements given in Section 5 contains most basic properties of complete residuated lattices, described in such a way as to show which conditions on the operation $\to$ allow them to be true. In particular it has been clearly shown the relevant role of commutativity.

- In case when the equivalence $\top \to x \geqslant a \Leftrightarrow x \geqslant a$ holds, $(M, \otimes, \top)$ is a *semi-norm* in the sense of [12] (i.e. $\otimes$ is isotonic in both variables, $\top$ is a unit and $\bot$ is a zero).
- Assuming that the condition $\top \to x \geqslant a \Rightarrow x \geqslant a$ holds, one gets that $(M, \otimes, \top)$ is a *complete quasi-monoidal lattice* [25] (i.e. a po-groupoid satisfying the conditions $a \leqslant a \otimes \top$ and $a \leqslant \top \otimes a$). In this case, as for cl-groupoids, one can extend the terminology of [25] and say that $(M, \otimes, \top)$ is a *right cl-quasi-monoid*.
- If $(M, \to, \top)$ is associative, then, once more extending notation of [25], one can say that $(M, \otimes, \top)$ is a *strictly right-sided right quantale* (i.e. $\otimes$ is associative and isotonic in both arguments, distributes over arbitrary $\bigvee$ on the right side and $\top$ is a right unit). Of course, if in this last case one adds the commutativity condition, i.e. the weak exchange condition of $\to$, then $(M, \otimes, \top)$ becomes a *strictly two sided commutative quantale* (see [25]) which is, evidently, nothing but a complete residuated lattice or, equivalently, an associative commutative cdeo algebra.

6. Concluding remarks and examples

One of the main motivation of this paper has been the investigation of the strict connection between the concept of order relation and the algebraic structures frequently used in many-valued logics and many-valued mathematics, in particular those provided by the connectives of implication and conjunction. The basic idea of looking at implication operator as nothing but the many-valued version of an order relation has given a tool for understanding the mathematical character and the role of the basic properties of implicative structures; as already remarked in the introduction our weak extended-order algebras are in fact implicative algebras in the terminology of [32].

We have not been directly interested in application to deductive systems or propositional calculi or other main topics of mathematical logic; instead, we have specialized our basic structure (w-eo algebra) from an algebraic point of view leading to lattice-ordered algebras most commonly used in many-valued mathematics and logics.

The comparison of these kind of algebras with some kinds of implicative structures more closely related to logic is a matter of forthcoming researches, where the MacNeille completion of implication algebras is also considered. Here we can remark that the representation theorem for positive implication algebras together with the results described in the previous sections show that those cdeo algebras that are positive implication algebras too are exactly the complete Heyting algebras.

Consequently one can argue that deo algebras that induce in their completion an adjoint product (resembling a conjunction connective) $\otimes \neq \wedge$ cannot be Hilbert algebras.

We can further note that if the MacNeille completion of an Hilbert algebra $(L, \to, \top)$ is still an Hilbert algebra then necessarily $(L, \to, \top)$ is an associative and commutative deo algebra. In the following example we exhibit an Hilbert algebra $(L, \to, \top)$ whose MacNeille completion $(K, \to_K, \top_K)$ is still an Hilbert algebra: of course, $(L, \to, \top)$ is an associative, commutative deo algebra and $(K, \to_K, \top_K)$ is a Heyting algebra.

Example 45. Let us consider the set $L = \{a, b, c, d, \top\}$ and the binary operation $\to$ be defined as follows:

$\to$	a	b	c	d	$\top$
a	$\top$	b	$\top$	$\top$	$\top$
b	a	$\top$	$\top$	$\top$	$\top$
c	a	b	$\top$	d	$\top$
d	a	b	c	$\top$	$\top$
$\top$	a	b	c	d	$\top$

Then $(L, \to, \top)$ is a Hilbert algebra whose MacNeille completion has support $K = \{\bot_K, \alpha, \beta, \gamma, \delta, \epsilon, \top_K\}$, where the elements of the completion are:

- $\bot_K = [L]$;
- $\alpha = [a]$;
- $\beta = [b]$;
- $\gamma = [c]$;
- $\delta = [d]$;
- $\epsilon = [\{c, d\}]$;
- $\top_K = [\top]$;

and $\to_K$ is defined by the following table:

$\to_K$	$\bot_K$	α	β	γ	δ	ϵ	$\top_K$
$\bot_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$
α	β	$\top_K$	β	$\top_K$	$\top_K$	$\top_K$	$\top_K$
β	α	α	$\top_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$
γ	$\bot_K$	α	β	$\top_K$	δ	δ	$\top_K$
δ	$\bot_K$	α	β	γ	$\top_K$	γ	$\top_K$
ϵ	$\bot_K$	α	β	$\top_K$	$\top_K$	$\top_K$	$\top_K$
$\top_K$	$\bot_K$	α	β	γ	δ	ϵ	$\top_K$

As for the multiplicative structures given by the conjunction connective or, more generally, connected with order relation, it is clear that the adjunction between conjunction and implication has inspired many investigations, concepts and results: mostly deriving an "implicative" operation (usually called residuum) from a suitable multiplication, distributive over joins, but also sometimes going in the converse direction. In [14] both the "residuation" and "deresiduation" processes are considered, within the universe of t-norms and their generalizations (mainly in the complete lattice [0, 1]); a few results about the characterization of commutative and associative multiplication in terms of the associated residuum overlap with the present paper. In [26] pseudo-BCK algebras are defined by a pair of implications $\to$ and $\rightsquigarrow$ so giving a somehow symmetrical character to the structure, that is commutative when the two implications coincide.

Our main effort, in this paper, has been to show when and how a quite general extension of an order relation can get the fundamental character of completeness giving a tool for the "deresiduation" process, without loosing its own properties. This has led to the notion of deo algebras which, as it is shown in Section 4, can be embedded into cdeo algebras; these form a class that includes most multiplicative structures connected with order and lattices (in the complete case) as it has been remarked in Section 5.

We conclude our discussion with some remarks on L-order relations. Of course there is a strict connection between eo algebras and the notion of L-order on a set X considered in [3,5,8,33]; such a many-valued order notion is given by means of a structure (complete residuated lattice) that gives an "internal many-valued order" in the range set of truth values. In the classical two-valued case this structure is given by the only operation that extends, in the sense we have specified in the introduction, the only (up to isomorphisms) order relation on the set {0, 1}, with the top element as truth value.

The structure of an eo algebra can be viewed as an "internal many-valued order" that is a binary operation that extends an order relation with a top element. So it seems to be quite natural to consider the definition of L-order assuming L to be an order algebra, possibly distributive and associative: this last condition would allow the transitivity condition of L-order to be expressed by means of the (strong) isotonic condition.

However the commutativity of L seems to be not a fundamental condition. In fact, after noting that the operation $\to$ of a (not necessarily complete) associative deo algebra satisfy reflexivity, antisymmetry and transitivity in the form of (strong) isotonic condition, one can see that the commutativity condition adds nothing but the possibility of expressing transitivity in the form of (strong) antitonic condition, too (see Corollary 44).

Forthcoming papers will be devoted to the investigation of behavior of powerset operators and Galois connections associated to binary L-relations, with L a (suitable) eo algebra, so applying the results of the present paper to also extend some results given in [2,16,17] for binary relations taking values in a complete residuated lattice; also, a many-valued approach to category theory based on eo algebras is being developed (see [20]). These investigations seem to enlighten quite well the role of associativity and commutativity conditions of a deo algebra.

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